弾性力学　　2020/06/25　　10:30〜

音声をオフにしてお待ち下さい。

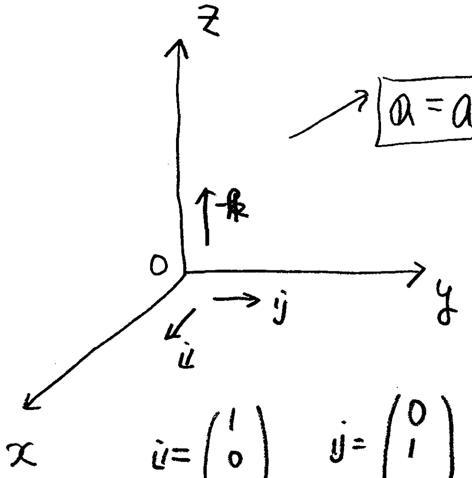
ベクトル（指標表示）

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

$$= a_1\begin{pmatrix}1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$$

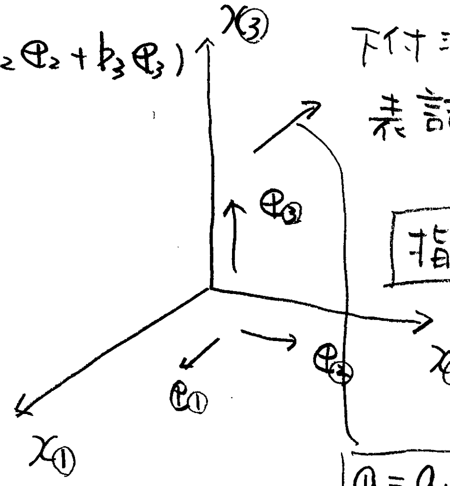
$\boxed{\text{マトリックス表示}}$



$\boxed{\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}}$

$$\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix},\quad \mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix},\quad \mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\Downarrow$



下付添字による表記法のこと

$\Downarrow$

$\boxed{\text{指標表示}}$

$\boxed{\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3}$

$|\mathbf{a}||\mathbf{b}|\cos\theta$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\cdot(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)$$

$$= (a_1\ a_2\ a_3)\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$\Downarrow$

$$\mathbf{e}_1\cdot\mathbf{e}_2 = (1\ 0\ 0)\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$$

$$\mathbf{e}_1\cdot\mathbf{e}_1 = (1\ 0\ 0)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$$

$\Downarrow$

$$\mathbf{a}\cdot\mathbf{b} = \boxed{\sum_{i=1}^{3}} a_i b_i = a_i b_i \Rightarrow \mathbf{a} = a_i\mathbf{e}_i = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

$$\mathbf{b} = b_j\mathbf{e}_j = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$$

省いてよい　　$\boxed{\text{総和規約}}$

添字のつけ替え（総和規約時）

$$a_i b_i = a_j b_j = a_k b_k = a_\alpha b_\alpha$$

外積

$\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}$, $\mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{0}$

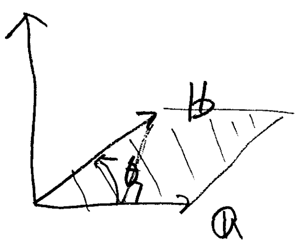
$\mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0}$, $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$

$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$, $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$

$\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3$, $\mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1$

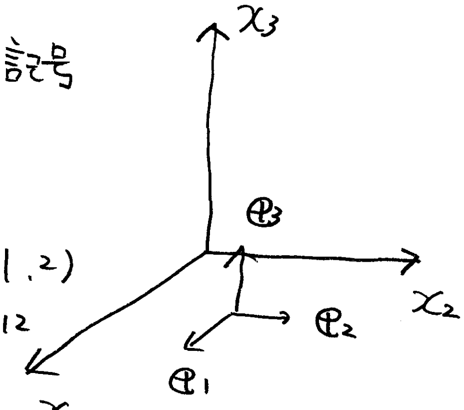
$\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2$

$\mathbf{a} \times \mathbf{b} \Rightarrow |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.



レビ・チビタ記号

$$\boxed{\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k}$$



$$\varepsilon_{ijk} = +1 : (1,2,3)\ (2,3,1),\ (3,1,2)$$

$$\varepsilon_{123} \quad \varepsilon_{231} \quad \varepsilon_{312}$$

$$= -1 : (3,2,1)\ (2,1,3)\ (1,3,2)$$

$$= 0 \quad \text{(2つ以上同じ添字)}$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = \varepsilon_{12k}\mathbf{e}_k = \varepsilon_{123}\mathbf{e}_3 = \mathbf{e}_3$$

$$\mathbf{a} \times \mathbf{b} = \begin{array}{c} \\ a_3 \\ b_3 \end{array}\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3.$$

$$= \boxed{\varepsilon_{ijk} a_i b_j \mathbf{e}_k}$$

$$= \underset{(+1)}{\varepsilon_{123}} a_1 b_2 \mathbf{e}_3 + \underset{(-1)}{\varepsilon_{132}} a_1 b_3 \mathbf{e}_2 + \underset{(-1)}{\varepsilon_{213}} a_2 b_1 \mathbf{e}_3 + \underset{(+1)}{\varepsilon_{231}} a_2 b_3 \mathbf{e}_1$$

$$+ \underset{(+1)}{\varepsilon_{312}} a_3 b_1 \mathbf{e}_2 + \underset{(-1)}{\varepsilon_{321}} a_3 b_2 \mathbf{e}_1$$

$$= (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3.$$

クロネッカーのデルタ

$$\boxed{\begin{aligned} \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} &= 1 \quad (i=j) \\ &= 0 \quad (i \neq j) \end{aligned}}$$

(性質)

$$\delta_{ij} = \delta_{ji}$$

$$\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

$$\delta_{ij} a_j = a_i$$

(解説) $\delta_{ij} a_j = \underbrace{\delta_{11}}_{+1} a_1 + \underbrace{\delta_{12}}_{0} a_2 + \underbrace{\delta_{13}}_{0} a_3 = a_1$

ε_{ijk} と δ_{ij} の関係

$$\boxed{\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}}$$

総和

(公式) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

$$\begin{aligned} (\varepsilon_{ijk} a_i b_j \boxed{\mathbf{e}_k) \times c_l \mathbf{e}_l} &= \varepsilon_{ijk} a_i b_j c_l \varepsilon_{klm} \mathbf{e}_m \\ &= \varepsilon_{ijk} \varepsilon_{klm} a_i b_j c_l \mathbf{e}_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_i b_j c_l \mathbf{e}_m \\ &= \underbrace{a_i c_i}_{(\mathbf{a}\cdot\mathbf{c})} \underbrace{b_j \mathbf{e}_j}_{\mathbf{b}} - \underbrace{b_j c_j}_{(\mathbf{b}\cdot\mathbf{c})} \underbrace{a_i \mathbf{e}_i}_{\mathbf{a}} \\ &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}. \end{aligned}$$

$$\boxed{\begin{aligned} \delta_{il} c_l &= c_i \\ \delta_{jm} \mathbf{e}_m &= \mathbf{e}_j \end{aligned}}$$

テンソル ⇒ 正方行列

$$\mathbf{e}_1 \odot \mathbf{e}_2 = \underset{1\times 3}{\begin{pmatrix}1 & 0 & 0\end{pmatrix}} \underset{3\times 1}{\begin{pmatrix}1\\0\\0\end{pmatrix}} = \underbrace{(\mathbf{e}_1)^T(\mathbf{e}_2)}_{1\times 1}$$

↑ 内積

$$\mathbf{e}_1 \otimes \mathbf{e}_2 = \underset{3\times 1}{\begin{pmatrix}1\\0\\0\end{pmatrix}} \underset{1\times 3}{\begin{pmatrix}0 & 1 & 0\end{pmatrix}} = \underbrace{\begin{pmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}}_{3\times 3}$$

↑ テンソル積

$$\mathbb{A} = [A_{ij}] = \begin{pmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{pmatrix}$$

$$= A_{11}\underbrace{\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}}_{\mathbf{e}_1\otimes\mathbf{e}_1} + A_{12}\underbrace{\begin{pmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}}_{\mathbf{e}_1\otimes\mathbf{e}_2} + \cdots\cdots A_{33}\underbrace{\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}}_{\mathbf{e}_3\otimes\mathbf{e}_3}$$

(行列)基底 → $\mathbf{e}_1\otimes\mathbf{e}_1$

$$= A_{ij}\,\mathbf{e}_i\otimes\mathbf{e}_j$$

$$\mathbb{A} = \mathbf{a}\otimes\mathbf{b} = a_i\,\mathbf{e}_i \otimes b_j\,\mathbf{e}_j = a_i b_j\,\mathbf{e}_i\otimes\mathbf{e}_j$$

$$= \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\begin{pmatrix}b_1 & b_2 & b_3\end{pmatrix} = \begin{pmatrix}a_1b_1 & a_1b_2 & a_1b_3\\a_2b_1 & a_2b_2 & a_2b_3\\a_3b_1 & a_3b_2 & a_3b_3\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{pmatrix}}_{\text{行列}}\underbrace{\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix}}_{\text{ベクトル}} = \begin{pmatrix}A_{11}C_1 + A_{12}C_2 + A_{13}C_3\\A_{21}C_1 + A_{22}C_2 + A_{23}C_3\\A_{31}C_1 + A_{32}C_2 + A_{33}C_3\end{pmatrix} = \begin{pmatrix}A_{1j}C_j\\A_{2j}C_j\\A_{3j}C_j\end{pmatrix}$$

$$\mathbb{A} = A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j , \qquad \mathbb{C} = C_l\, \mathbf{e}_l$$

$$\longrightarrow \quad \mathbb{A} \cdot \mathbb{C} = (A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j) \cdot C_l\, \mathbf{e}_l = A_{ij} C_j\, \mathbf{e}_i$$

$$\delta_{jl}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= A_{1j} C_j\, \mathbf{e}_1 + A_{2j} C_j\, \mathbf{e}_2 + A_{3j} C_j\, \mathbf{e}_3$$

$$= \begin{pmatrix} A_{1j} C_j \\ A_{2j} C_j \\ A_{3j} C_j \end{pmatrix} \longleftarrow \text{先ほどと一致}$$

$$\begin{pmatrix} C_1 & C_2 & C_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{A_{i1} C_i}_{A_{11}C_1 + A_{21}C_2 + A_{31}C_3} & A_{i2} C_i & A_{i3} C_i \end{pmatrix}$$

$$\mathbb{C} \cdot \mathbb{A} = C_l\, \mathbf{e}_l \cdot (A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j) = C_i A_{ij}\, \mathbf{e}_j$$

$$\delta_{li}$$

行ベクトル

$$A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = \begin{pmatrix} A_{1k}B_{k1} & A_{1k}B_{k2} & A_{1k}B_{k3} \\ A_{2k}B_{k1} & A_{2k}B_{k2} & A_{2k}B_{k3} \\ A_{3k}B_{k1} & A_{3k}B_{k2} & A_{3k}B_{k3} \end{pmatrix}$$

$$\Updownarrow$$

$$\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}$$

$$\mathbb{A} \cdot \mathbb{B} = (A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j) \cdot (B_{kl}\, \mathbf{e}_k \otimes \mathbf{e}_l) = A_{ik} B_{kl}\, \mathbf{e}_i \otimes \mathbf{e}_l$$

$$\delta_{jk}$$

$$\mathbb{A}^T = A_{ij}\, \mathbf{e}_j \otimes \mathbf{e}_i = A_{ji}\, \mathbf{e}_j \otimes \mathbf{e}_i$$

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ベクトル場・テンソル場の微分・積分

(ナブラ) $\nabla = \nabla_i e_i = \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} \overset{e_3}{} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}$

$\varphi(x_1, x_2, x_3)$

(勾配) $\nabla \varphi = \frac{\partial \varphi}{\partial x_1} e_1 + \frac{\partial \varphi}{\partial x_2} e_2 + \frac{\partial \varphi}{\partial x_3} e_3 = \begin{pmatrix} \frac{\partial \varphi}{\partial x_1} \\ \frac{\partial \varphi}{\partial x_2} \\ \frac{\partial \varphi}{\partial x_3} \end{pmatrix}$

δ_{ij}

(発散) $\nabla \cdot a = \nabla_i \boxed{e_i \cdot a_j e_j} = \nabla_i a_i$

$= \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$　↑数学I

⇒ ベクトルだけではベクトルの勾配が扱えない

$\vec{\nabla} \otimes a = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} (a_1 \ a_2 \ a_3) = \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_2}{\partial x_1} & \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_1}{\partial x_2} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \\ \frac{\partial a_1}{\partial x_3} & \frac{\partial a_2}{\partial x_3} & \frac{\partial a_3}{\partial x_3} \end{pmatrix}$

あまりつかわない

$= \frac{\partial a_1}{\partial x_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{e_1 \otimes e_1} + \frac{\partial a_2}{\partial x_1} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{e_1 \otimes e_2} + \cdots + \frac{\partial a_3}{\partial x_3} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{e_3 \otimes e_3}$

$= \frac{\partial a_j}{\partial x_i} e_i \otimes e_j$

$$a \otimes \overleftarrow{\nabla} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{pmatrix}$$

$$= \frac{\partial a_1}{\partial x_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{e_1 \otimes e_1} + \frac{\partial a_1}{\partial x_2} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{e_1 \otimes e_2} + \cdots + \frac{\partial a_3}{\partial x_3} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{e_3 \otimes e_3}$$

$$= \frac{\partial a_i}{\partial x_j} e_i \otimes e_j$$

$$\underbrace{\vec{\nabla} \cdot a = a \cdot \overleftarrow{\nabla}}_{\nabla \cdot a} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$= \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$$

$$\vec{\nabla} \cdot A = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$\underbrace{\frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{21}}{\partial x_2} + \frac{\partial A_{31}}{\partial x_3}}$$

$$= \begin{pmatrix} \underbrace{\frac{\partial A_{i1}}{\partial x_i}}_{e_1} & \underbrace{\frac{\partial A_{i2}}{\partial x_i}}_{e_2} & \underbrace{\frac{\partial A_{i3}}{\partial x_i}}_{e_3} \end{pmatrix}$$

$$= \frac{\partial A_{ij}}{\partial x_i} e_j \quad //$$

$$\boxed{\vec{\nabla} \cdot A = \frac{\partial}{\partial x_k} \underbrace{e_k \cdot (A_{ij}\, e_i}_{\delta_{ki}} \otimes e_j) = \frac{\partial A_{ij}}{\partial x_i} e_j}$$

$$\mathbb{A}\cdot\overleftarrow{\nabla} = \begin{array}{cc}\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} & \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \\ \text{行列} & \text{ベクトル} \end{array} = \begin{pmatrix} \frac{\partial A_{1j}}{\partial x_j} \\ \frac{\partial A_{2j}}{\partial x_j} \\ \frac{\partial A_{3j}}{\partial x_j} \end{pmatrix} = \frac{\partial A_{ij}}{\partial x_j}\mathbb{e}_i$$

$$\boxed{\mathbb{A}\cdot\overleftarrow{\nabla} = (A_{ij}\,\mathbb{e}_i\otimes\mathbb{e}_j)\cdot\left(\frac{\partial}{\partial x_k}\mathbb{e}_k\right) = \frac{\partial A_{ij}}{\partial x_j}\mathbb{e}_i \qquad \delta_{jk}}$$

発散定理

数学I →

$$\boxed{\begin{aligned}\int\vec{\nabla}\cdot\mathbb{a}\,dV &= \int\left(\frac{\partial}{\partial x_1}\ \frac{\partial}{\partial x_2}\ \frac{\partial}{\partial x_3}\right)\begin{pmatrix} a_1 \\ a_2 \\ a_3\end{pmatrix} dV \\ &\qquad \frac{\partial}{\partial x_1}\to n_1,\ \frac{\partial}{\partial x_2}\to n_2,\ \frac{\partial}{\partial x_3}\to n_3,\ dV\to dS \\ &= \int(n_1\ n_2\ n_3)\begin{pmatrix} a_1 \\ a_2 \\ a_3\end{pmatrix} dS \\ &= \int n_i a_i\, dS \\ &= \int \mathbb{n}\cdot\mathbb{a}\, dS\end{aligned}}$$

$$\underbrace{\int\vec{\nabla}\cdot\mathbb{A}\,dV}_{\frac{\partial A_{ij}}{\partial x_i}\mathbb{e}_j} = \int\left(\frac{\partial}{\partial x_1}\ \frac{\partial}{\partial x_2}\ \frac{\partial}{\partial x_3}\right)\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} dV$$

$$\frac{\partial}{\partial x_1}\to n_1,\quad \frac{\partial}{\partial x_2}\to n_2,\quad \frac{\partial}{\partial x_3}\to n_3$$

$$= \int n_i A_{ij}\,\mathbb{e}_j\, dS$$

$$\to \left(\int \underbrace{\frac{\partial A_{ij}}{\partial x_i}dV}\right)\mathbb{e}_j = \left(\int \underbrace{n_i A_{ij}\,dS}\right)\mathbb{e}_j$$

基底をおとす

$$\to \int\frac{\partial A_{ij}}{\partial x_i}dV = \int n_i A_{ij}\,dS$$

$$\int \underset{\sim\sim\sim}{A \cdot \nabla}\, dV = \int \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} dV$$

$$\frac{\partial A_{ij}}{\partial x_j} e_i$$

$$= \int \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} dS$$

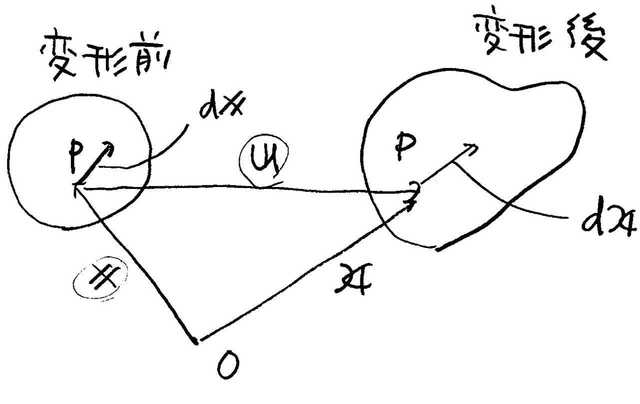
$$= \int A_{ij} n_j e_i \, dS$$

$$\longrightarrow \int \left(\frac{\partial A_{ij}}{\partial x_j} e_i \right) dV = \int (A_{ij} n_j e_i) \, dS$$

$$\longrightarrow \int \frac{\partial A_{ij}}{\partial x_j} dV = \int A_{ij} n_j \, dS$$

ベクトル・テンソルの説明終了

変形勾配テンソル



$$x = X + u \longrightarrow x_i = X_i + u_i \qquad (i = 1, 2, 3)$$

$$(\text{全微分}) \quad dx_i = \frac{\partial x_i}{\partial X_1} dX_1 + \frac{\partial x_i}{\partial X_2} dX_2 + \frac{\partial x_i}{\partial X_3} dX_3$$

$$= \frac{\partial x_i}{\partial X_j} dX_j$$

変形勾配テンソル

$$\Longrightarrow dx_i = \frac{\partial x_i}{\partial X_j} dX_j \Rightarrow \begin{pmatrix} dx_1 \\ dx_2 \\ dx_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} \begin{pmatrix} dX_1 \\ dX_2 \\ dX_3 \end{pmatrix}$$

$$\underbrace{\mathbb{F}}_{\text{変形勾配テンソル}} = F_{ij}\,\mathbb{e}_i \otimes \mathbb{e}_j = \frac{\partial x_i}{\partial X_j}\,\mathbb{e}_i \otimes \mathbb{e}_j$$

$x = X + u$

$$= \frac{\partial (X_i + u_i)}{\partial X_j}\,\mathbb{e}_i \otimes \mathbb{e}_j$$

$$= \Big(\underbrace{\frac{\partial X_i}{\partial X_j}}_{\delta_{ij}} + \frac{\partial u_i}{\partial X_j}\Big)\,\mathbb{e}_i \otimes \mathbb{e}_j$$

直接表記

$$= \big(\underbrace{\delta_{ij}}_{\text{対角行列}} + \frac{\partial u_i}{\partial X_j}\big)\,\mathbb{e}_i \otimes \mathbb{e}_j = \boxed{\mathbb{I} + \mathbb{u} \otimes \nabla}$$

$u_1 + X_1$

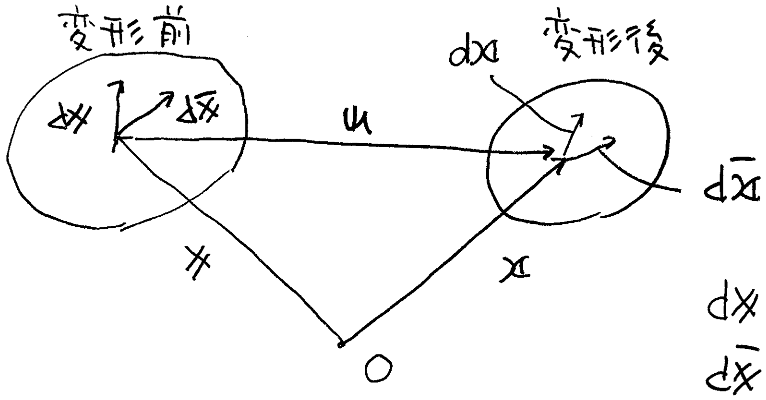
$$\underbrace{[F_{ij}]}_{\text{行列}} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial (X_1+u_1)}{\partial X_1} & \frac{\partial (X_1+u_1)}{\partial X_2} & \frac{\partial (X_1+u_1)}{\partial X_3} \\ \frac{\partial (X_2+u_2)}{\partial X_1} & \frac{\partial (X_2+u_2)}{\partial X_2} & \frac{\partial (X_2+u_2)}{\partial X_3} \\ \frac{\partial (X_3+u_3)}{\partial X_1} & \frac{\partial (X_3+u_3)}{\partial X_2} & \frac{\partial (X_3+u_3)}{\partial X_3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & 1 + \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & 1 + \frac{\partial u_3}{\partial X_3} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{[\delta_{ij}]} + \underbrace{\begin{pmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{pmatrix}}_{\left[\frac{\partial u_i}{\partial X_j}\right]}$$

弾性力学　2020/07/02　10:50〜

※ 7/6 は休講にします。



$dX \to dx$

$d\bar{X} \to d\bar{x}$

変形勾配テンソル

$$dx_i = \underbrace{F_{ij}}\, dX_j = \left(\underbrace{\delta_{ij} + \frac{\partial u_i}{\partial X_j}}\right) dX_j$$

$$d\bar{x}_i = F_{ij}\, d\bar{X}_j = \left(\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right) d\bar{X}_j$$

変形前後の内積差

$$dx \cdot d\bar{x} - dX \cdot d\bar{X} = dx_k\, d\bar{x}_k - dX_k\, d\bar{X}_k$$

$$= \left(\delta_{ki} + \frac{\partial u_k}{\partial X_i}\right) dX_i \left(\delta_{kj} + \frac{\partial u_k}{\partial X_j}\right) d\bar{X}_j - dX_k\, d\bar{X}_k$$

$$= \left(dX_k + \frac{\partial u_k}{\partial X_i} dX_i\right)\left(d\bar{X}_k + \frac{\partial u_k}{\partial X_j} d\bar{X}_j\right) - dX_k\, d\bar{X}_k$$

$$= \underbrace{\frac{\partial u_k}{\partial X_j} dX_k\, d\bar{X}_j}_{k \to i} + \underbrace{\frac{\partial u_k}{\partial X_i} dX_i\, d\bar{X}_k}_{k \to j} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} dX_i\, d\bar{X}_j$$

$$= \left(\boxed{\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}}\right) dX_i\, d\bar{X}_j$$

$$E_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \underbrace{\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}}_{\text{幾何学的非線形}}\right)$$

（グリーンのひずみテンソル）

幾何学的非線形 → 座屈、塑性

- $\dfrac{\partial u_k}{\partial x_i}\dfrac{\partial u_k}{\partial x_j} \cong 0$ ← $\boxed{\dfrac{\partial u_k}{\partial x_i} \fallingdotseq 0}$

- $\dfrac{\partial}{\partial x_j} = \underbrace{\boxed{\dfrac{\partial x_1}{\partial x_j}}\dfrac{\partial}{\partial x_1} + \boxed{\dfrac{\partial x_2}{\partial x_j}}\dfrac{\partial}{\partial x_2} + \boxed{\dfrac{\partial x_3}{\partial x_j}}\dfrac{\partial}{\partial x_3}}_{1 \text{ or } 0}$

$$= \frac{\partial(x_1+u_1)}{\partial x_j}\frac{\partial}{\partial x_1} + \frac{\partial(x_2+u_2)}{\partial x_j} + \frac{\partial(x_3+u_3)}{\partial x_j}\frac{\partial}{\partial x_3}$$

$$= \delta_{ij}\frac{\partial}{\partial x_i} \quad \left(= \underbrace{\frac{\partial x_1}{\partial x_j}}_{\delta_{1j}}\frac{\partial}{\partial x_1} + \underbrace{\frac{\partial x_2}{\partial x_j}}_{\delta_{2j}}\frac{\partial}{\partial x_2} + \underbrace{\frac{\partial x_3}{\partial x_j}}_{\delta_{3j}}\frac{\partial}{\partial x_3} \right)$$

$$= \frac{\partial}{\partial x_j}$$

$\Rightarrow \boxed{\varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)}$ （弾性力学のひずみ）

$$\Rightarrow [\varepsilon_{ij}] = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3}+\dfrac{\partial u_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3}+\dfrac{\partial u_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_1}+\dfrac{\partial u_1}{\partial x_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_2}+\dfrac{\partial u_2}{\partial x_3}\right) & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}$$

せん断ひずみ

垂直ひずみ

対称性がある

$u_1 \to u,\ u_2 \to v,\ u_3 \to w$
$x_1 \to x,\ x_2 \to y,\ x_3 \to z$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\partial w}{\partial z}$$

$$\gamma_{12} = \varepsilon_{12} + \varepsilon_{21} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{13} = \varepsilon_{13} + \varepsilon_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\gamma_{23} = \varepsilon_{23} + \varepsilon_{32} = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

↑ 工学ひずみ

σ_{ij}（応力）と ε_{ij}（ひずみ）の関係（フックの法則） $\left(\text{材力}\ \begin{array}{l} \boxed{\sigma} = E\varepsilon \\ \boxed{\tau} = G\gamma \end{array}\right)$

応力　ひずみ

垂直ひずみ

$$\begin{cases} \varepsilon_{11} = \dfrac{\sigma_{11}}{E} - \nu\left(\dfrac{\sigma_{22}}{E} + \dfrac{\sigma_{33}}{E}\right) \\ \varepsilon_{22} = \dfrac{\sigma_{22}}{E} - \nu\left(\dfrac{\sigma_{11}}{E} + \dfrac{\sigma_{33}}{E}\right) \\ \varepsilon_{33} = \dfrac{\sigma_{33}}{E} - \nu\left(\dfrac{\sigma_{11}}{E} + \dfrac{\sigma_{22}}{E}\right) \end{cases}$$

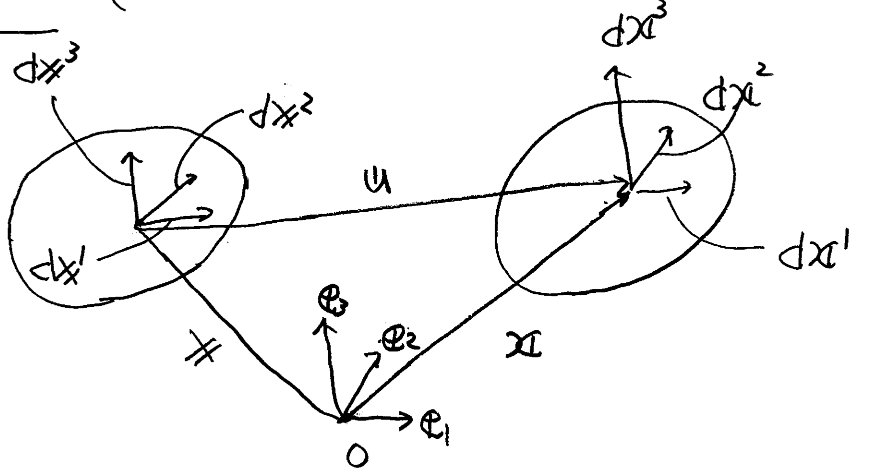
せん断ひずみ

$$\begin{cases} \gamma_{23} = \dfrac{\sigma_{23}}{G} & \tau_{23} \\ \gamma_{31} = \dfrac{\sigma_{31}}{G} & \tau_{31} \\ \gamma_{12} = \dfrac{\sigma_{12}}{G} & \tau_{12} \end{cases}$$

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix}}_{\text{コンプライアンス行列}} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}$$

$$\rightleftarrows \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \underbrace{\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \begin{pmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & & & \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & & & \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & & & \\ & & & \frac{1-2\nu}{2(1-\nu)} & & \\ & & & & \frac{1-2\nu}{2(1-\nu)} & \\ & & & & & \frac{1-2\nu}{2(1-\nu)} \end{pmatrix}}_{\text{剛性行列} = (\text{コンプライアンス行列})^{-1}} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix}$$

体積ひずみ（圧縮性 or 非圧縮）



$$d\boldsymbol{X}^1 = ds^1\boldsymbol{e}_1,\quad d\boldsymbol{X}^2 = ds^2\boldsymbol{e}_2,\quad d\boldsymbol{X}^3 = ds^3\boldsymbol{e}_3$$

$$dV_0 = d\boldsymbol{X}^1\cdot(d\boldsymbol{X}^2\times d\boldsymbol{X}^3) = ds^1ds^2ds^3$$

$$\left(= (ds^1ds^2ds^3)\big(\underbrace{\boldsymbol{e}_1\cdot(\boldsymbol{e}_2\times\boldsymbol{e}_3)}_{1}\big)\right)$$

$$d\boldsymbol{x}^1 = \mathbb{F}\cdot d\boldsymbol{X}^1 = (F_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot ds^1\boldsymbol{e}_1$$

δ_{j1}

$$= F_{i1}\boldsymbol{e}_i\,ds^1$$

i

$$d\boldsymbol{x}^i = F_{ji}\boldsymbol{e}_j\,ds^i$$

$$= \begin{pmatrix} F_{1i}ds^i \\ F_{2i}ds^i \\ F_{3i}ds^i \end{pmatrix}$$

$$dV = d\boldsymbol{x}^1\cdot(d\boldsymbol{x}^2\times d\boldsymbol{x}^3) = \begin{vmatrix} F_{11}ds^1 & F_{12}ds^2 & F_{13}ds^3 \\ F_{21}ds^2 & F_{22}ds^2 & F_{23}ds^3 \\ F_{31}ds^3 & F_{32}ds^2 & F_{33}ds^3 \end{vmatrix}$$

$$= ds^1ds^2ds^3\det(F_{ij})$$

$$J = \frac{dv}{dV_0} = \det(F_{ij}) = \begin{vmatrix} 1+\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & 1+\frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & 1+\frac{\partial u_3}{\partial X_3} \end{vmatrix}$$

$$\cong 1 + \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_3}$$

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial X_i}$$

$$= 1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$e = \frac{dv - dV_0}{dV_0} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

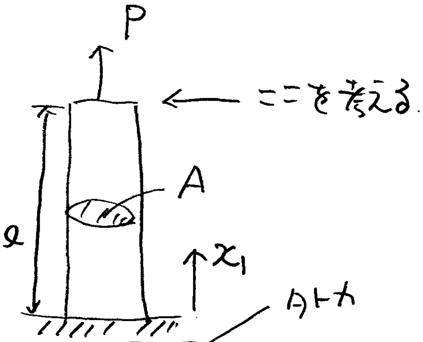
<弾性力学>

2020/07/09

応力

$$\boxed{n_i \sigma_{ij} = t_j}$$ (コーシーの公式) (9.4)

→ 教科書 P.50〜52



外力

内力 $\boxed{\sigma = \frac{P}{A}}$ (材料力学) 1次元

$\boxed{\varepsilon = \frac{\sigma}{E}}$ (材料力学)

内力：変形に対する抵抗力(単位面積)

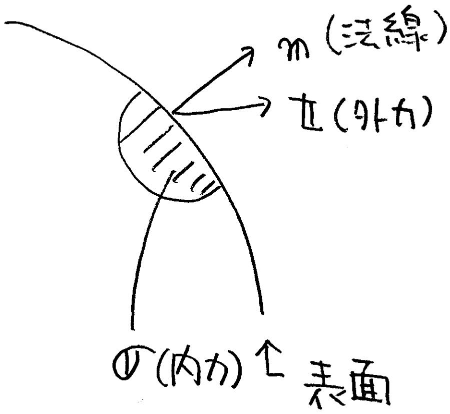
(一次元)

(単位面積)表面力

$$\underbrace{n_1}_{1}\underbrace{\sigma_{11}}_{\sigma} = \underbrace{t_1}_{P/A}$$

$$\rightarrow \sigma = \frac{P}{A}$$

(三次元)



$t_j e_j$

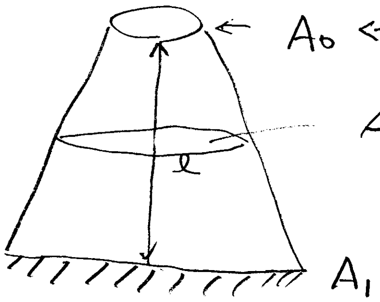
$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$$

$$= n_k \underbrace{e_k \cdot (\sigma_{ij}\, e_i}_{\delta_{ki}} \otimes e_j)$$

$$= n_i \sigma_{ij} e_j$$

$$\rightarrow \boxed{t_j = n_i \sigma_{ij}}$$

境界条件



$$\sigma = \frac{P}{A_0}$$

$$A = A_0 + \frac{A_1 - A_0}{l} x$$

$$\rightarrow \sigma = \frac{P}{A} \fallingdotseq \frac{P}{A_0}$$

(つり合いの式)

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \underbrace{\rho_0 k_j}_{\text{重力}} = 0 \qquad (12.9)$$

$\left(\rho \dfrac{Dv_j}{Dt}\right.$: 流体力学)

← 保存則から導出される

(一次元 + 重力が無視出来る)

$$\boxed{\frac{\partial \sigma_{11}}{\partial x_1} = 0} \quad + \quad \boxed{\sigma_{11}\Big|_{x_1 = \ell} = \frac{P}{A}}$$

つり合いの式 　　　　 コーシーの公式

↓

$$\sigma_{11} = C$$

↓

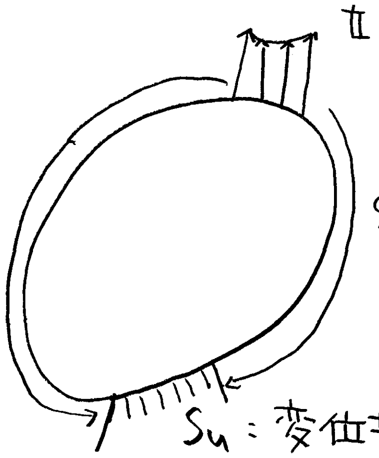
$$\boxed{\sigma_{11} = \frac{P}{A}}$$

材料力学の式（応力法）

<弾性力学>のまとめ

- $\dfrac{\partial \sigma_{ij}}{\partial x_i} = 0$ 　(つり合い) $\boxed{3}$
- $\varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$ (変位－ひずみ) $\boxed{1}$
- $$\boxed{\begin{aligned} \varepsilon_{11} &= \frac{\sigma_{11}}{E} - \nu\left(\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}\right) \\ \varepsilon_{22} &= \frac{\sigma_{22}}{E} - \nu\left(\frac{\sigma_{11}}{E} + \frac{\sigma_{33}}{E}\right) \\ \varepsilon_{33} &= \frac{\sigma_{33}}{E} - \nu\left(\frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E}\right) \\ \gamma_{23} &= \frac{\tau_{23}}{G}, \quad \gamma_{31} = \frac{\tau_{31}}{G}, \quad \gamma_{12} = \frac{\tau_{12}}{G} \end{aligned}}$$

(構成則) $\boxed{2}$



S_t : 力学的境界条件

S_u : 変位境界条件

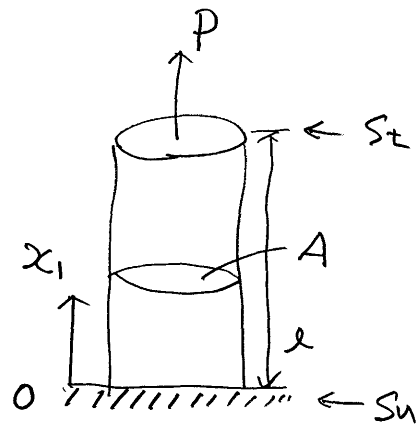
- $u_i = \underline{u}_i$ ← Given on S_u
- $n_i \sigma_{ij} = \underline{t}_j$ on S_t

(境界条件) $\boxed{4}$

$\boxed{1} \rightarrow \boxed{2} \rightarrow \boxed{3} \rightarrow \boxed{4}$ = ナビアの式（変位法）

$\boxed{1}$

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{du_1}{dx_1} + \frac{du_1}{dx_1}\right) = \frac{du_1}{dx_1}$$



$\boxed{2}$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E}$$

$\boxed{3}$

$$\frac{d\sigma_{11}}{dx_1} = 0$$

$\boxed{4}$

$$S_u : u_1 = 0 \quad @\ x_1 = 0 \leftarrow \boxed{4}-1$$

$$S_t : \sigma_{11} = \frac{P}{A} \quad @\ x_1 = l \leftarrow \boxed{4}-2$$

解法 : $\boxed{1} \rightarrow \boxed{2} \rightarrow \boxed{3}$

$$\sigma_{11} = E\varepsilon_{11} = E\frac{du_1}{dx_1}$$

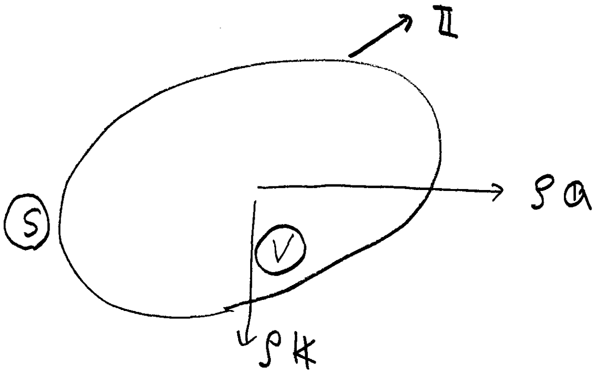
$$\rightarrow \frac{d\sigma_{11}}{dx_1} = E\frac{d^2u_1}{dx_1^2} = 0 \qquad \text{(ナビアの式)}$$

$$\rightarrow u_1 = C_1 x_1 + \underset{\substack{\parallel \\ 0\ (\because \boxed{4}-1)}}{C_2}$$

$$\rightarrow \underset{@\ x_1 = l}{\sigma_{11}} = \frac{P}{A}\ \boxed{\boxed{4}-2} \rightarrow E\underbrace{\frac{du_1}{dx_1}}_{C_1}\bigg|_{x_1=l} = \frac{P}{A} \rightarrow C_1 = \frac{P}{EA}$$

$$\rightarrow \therefore \boxed{u_1 = \frac{Px_1}{EA}} \rightarrow \boxed{\begin{aligned} \varepsilon_{11} &= \frac{du_1}{dx_1} = \frac{P}{EA} \\ \sigma_{11} &= E\varepsilon_{11} = \frac{P}{A} \end{aligned}}$$

つり合いの式



$$\boxed{\boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \cdot \boldsymbol{n}}$$

オイラーによる運動量保存則

$$\int_V \overset{m}{\boxed{\rho \boldsymbol{a}\, dV}} = \int_S \boldsymbol{T}\, ds + \int_V \overset{m}{\rho \boldsymbol{k}\, dV}$$

$$\boxed{\int_V \rho \boldsymbol{a}\, dV = \int_S (\boldsymbol{\sigma}^T \cdot \boldsymbol{n})\, ds + \int_V \rho \boldsymbol{k}\, dV}$$

$$\int_V \rho a_j\, dV = \int_S \sigma_{ij} n_i\, ds + \int_V \rho k_j\, dV$$

$$= \int_V \frac{\partial \sigma_{ij}}{\partial x_i}\, dV + \int_V \rho k_j\, dV$$

$$\rightarrow \int_V \underbrace{\left(\rho a_j - \frac{\partial \sigma_{ij}}{\partial x_i} - \rho k_j\right)}_{=0} dV = 0$$

$$\rightarrow \underset{0=}{\rho a_j} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho k_j$$

$$\rightarrow \boxed{\frac{\partial \sigma_{ij}}{\partial x_i} + \rho k_j = 0}$$

つり合いの式

$$\int_V \rho a\, dV = \int_S (\sigma^T \cdot n)\, ds + \int_V \rho k\, dV$$

$$\int_V \begin{pmatrix} \rho a_1 \\ \rho a_2 \\ \rho a_3 \end{pmatrix} dV = \int_{\underset{\downarrow}{S}} \begin{pmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \overset{\frac{\partial}{\partial x_1}}{n_1} \\ n_2 \\ n_3 \end{pmatrix} \underset{\downarrow}{ds} + \int_V \begin{pmatrix} \rho k_1 \\ \rho k_2 \\ \rho k_3 \end{pmatrix} dV$$

$$\qquad V \qquad\qquad \boxed{\text{ガウスの発散定理}} \qquad dV$$

$$\rightarrow \int_V \begin{pmatrix} \rho a_1 \\ \rho a_2 \\ \rho a_3 \end{pmatrix} dV = \int_V \begin{pmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} dV + \int_V \begin{pmatrix} \rho k_1 \\ \rho k_2 \\ \rho k_3 \end{pmatrix} dV$$

$$\rightarrow \int_V \underbrace{\begin{pmatrix} \rho a_1 \\ \rho a_2 \\ \rho a_3 \end{pmatrix}} dV = \int_V \underbrace{\begin{pmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{pmatrix}} dV$$

$$+ \int_V \underbrace{\begin{pmatrix} \rho k_1 \\ \rho k_2 \\ \rho k_3 \end{pmatrix}} dV$$

$$\rightarrow \begin{pmatrix} \rho a_1 \\ \rho a_2 \\ \rho a_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho k_1 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho k_2 \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho k_3 \end{pmatrix}$$

$$\rightarrow \overset{=0}{\rho a_j} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho k_j \rightarrow \boxed{\frac{\partial \sigma_{ij}}{\partial x_1} + \rho k_j = 0}$$

つり合いの式

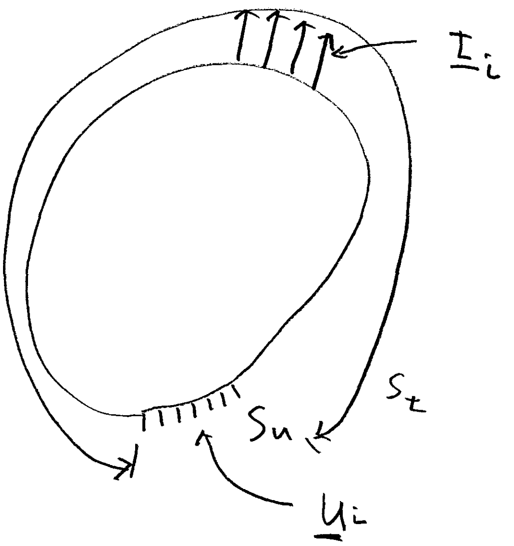
$$\int_V a \cdot \overset{\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}}{n}\, ds = \int_V \overset{\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}}{\nabla} \cdot a\, dV$$

弾性力学　　　2020/07/13　　10:30〜

<境界値問題>

(つり合い式)

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho k_j = 0$$



(変位－ひずみ関係式)

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

(応力－ひずみ関係式(構成則))

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$ (教科書 11.2参照)

(境界条件)

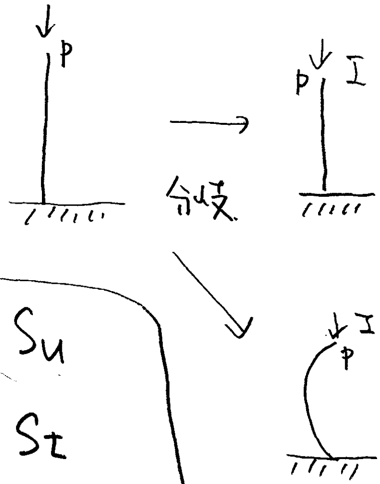
$$u_i = \underline{u}_i$$

$$\sigma_{ij} n_i = \underline{t}_j$$ (コーシーの公式)

<解の一意性>　(微小変形理論) → 座屈

2組の解を考える。

$(u_i', \varepsilon_{ij}', \sigma_{ij}')$, $(u_i'', \varepsilon_{ij}'', \sigma_{ij}'')$ (正解)



(試行関数)

$$\begin{cases} \hat{u}_i = u_i' - u_i'' \\ \hat{\varepsilon}_{ij} = \varepsilon_{ij}' - \varepsilon_{ij}'' \\ \hat{\sigma}_{ij} = \sigma_{ij}' - \sigma_{ij}'' \end{cases} \Rightarrow$$

$\hat{u}_i = 0$ on S_u

$n_i\hat{\sigma}_{ij} = 0$ on S_t

$$\frac{\partial \hat{\sigma}_{ij}}{\partial x_i} = 0$$

$$\hat{\sigma}_{ij} = \lambda\hat{\varepsilon}_{kk}\delta_{ij} + 2\mu\hat{\varepsilon}_{ij}$$

$$\hat{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}\right)$$

$$I = \int_{S} \hat{u}_j \hat{\sigma}_{ij} n_i \, dS = 0 \qquad \left(\begin{array}{l} \because \; S = S_u + S_t \\ \hat{\sigma}_{ij} n_i = 0 \quad \text{on } S_t \\ \hat{u}_j = 0 \quad \text{on } S_u \end{array} \right)$$

$$S \to V, \qquad n_i \to \frac{\partial}{\partial x_i}, \qquad dS \to dV$$

$$I = \int_V \frac{\partial}{\partial x_i} \left(\hat{u}_j \hat{\sigma}_{ij} \right) dV$$

$$= \int_V \left(\frac{\partial \hat{u}_j}{\partial x_i} \hat{\sigma}_{ij} + \underbrace{\frac{\partial \hat{\sigma}_{ij}}{\partial x_i}}_{=0} u_j \right) dV$$

$$= \int_V \frac{\partial \hat{u}_j}{\partial x_i} \hat{\sigma}_{ij} \, dV$$

$$\hat{\sigma}_{ij} = \frac{1}{2} \left(\hat{\sigma}_{ij} + \hat{\sigma}_{ji} \right)$$

$$= \int_V \left(\frac{1}{2} \frac{\partial \hat{u}_j}{\partial x_i} \hat{\sigma}_{ij} + \frac{1}{2} \underbrace{\frac{\partial \hat{u}_j}{\partial x_i} \hat{\sigma}_{ji}}_{j \rightleftarrows i} \right)$$

$$= \int_V \underbrace{\frac{1}{2} \left(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} \right)}_{\hat{\varepsilon}_{ij}} \hat{\sigma}_{ij} \, dV$$

$$\hat{\sigma}_{ij} = \lambda \hat{\varepsilon}_{kk} \delta_{ij} + 2\mu \hat{\varepsilon}_{ij} \qquad \alpha \qquad \alpha$$

$$\hat{\varepsilon}_{ii} \leftarrow \lambda \hat{\varepsilon}_{kk} \delta_{ij} \rightarrow \hat{\varepsilon}_{ij}$$

正

$$= \int_V \left(\lambda \hat{\varepsilon}_{kk}^2 + 2\mu \underline{\hat{\varepsilon}_{ij} \hat{\varepsilon}_{ij}} \right) dV$$

$$\hat{\varepsilon}_{kk}^2 \to \left(\hat{\varepsilon}_{11} + \hat{\varepsilon}_{22} + \hat{\varepsilon}_{33} \right)^2 \geq 0 \qquad \hat{\varepsilon}_{ij}\hat{\varepsilon}_{ij} \to \left(\hat{\varepsilon}_{11}^2 + \hat{\varepsilon}_{12}^2 + \cdots + \hat{\varepsilon}_{33}^2 \right) \geq 0$$

$$\sum_{k=1}^{3} \hat{\varepsilon}_{kk} = \hat{\varepsilon}_{kk}$$

$$= 0 \;\Rightarrow\; \hat{\varepsilon}_{ij} = 0 \;\Rightarrow\; \boxed{\hat{\varepsilon}_{ij} = \varepsilon'_{ij} - \varepsilon''_{ij} = 0}$$

$$\Downarrow$$

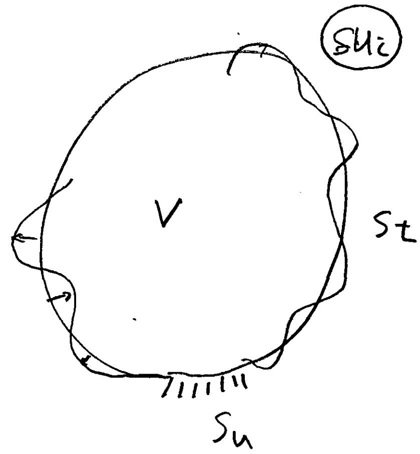
$$\hat{\sigma}_{ij} = 0 \;\Rightarrow\; \boxed{\hat{\sigma}_{ij} = \sigma'_{ij} - \sigma''_{ij} = 0}$$

$$\Downarrow$$

$$\boxed{\hat{u}_i = 0}$$

<仮想仕事の式>

$$\int_V \underset{\text{正解}}{\sigma_{ij}}\, \underset{\text{仮想}}{\delta\varepsilon_{ij}}\, dV - \int_V \overset{\text{動}}{\rho k_j}\, \underset{\text{仮想}}{\delta u_j}\, dV - \int_{S_t} \underset{\text{表面力}}{t_j}\, \underset{\text{仮想}}{\delta u_j}\, dS = 0$$



$$\delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right)$$

⇒ 有限要素法の基礎式

$$\underset{\text{便宜上}}{-}\int_V \Big(\underbrace{\frac{\partial \sigma_{ij}}{\partial x_i} + \rho k_j}_{=0}\Big)\delta u_j\, dV \quad \int_{S_t} \underbrace{(\sigma_{ij} n_i - t_j)}_{=0}\, \underset{S_u}{\delta u_j}\, dS = 0$$

$$-\int_V \left(\frac{\partial \sigma_{ij}}{\partial x_i} + \rho k_j\right)\delta u_j\, dV + \int_S \sigma_{ij} n_i\, \delta u_j\, dS - \int_{S_t} t_j\, \delta u_j\, dS = 0$$

$$S \to V,\quad n_i \to \frac{\partial}{\partial x_i},\quad dS \to dV$$

$$-\int_V \left(\cancel{\frac{\partial \sigma_{ij}}{\partial x_i}} + \rho k_j\right)\delta u_j\, dV + \int_V \frac{\partial}{\partial x_i}(\sigma_{ij}\,\delta u_j)\, dV - \int_{S_t} t_j\, \delta u_j\, dS = 0$$

$$\frac{\partial}{\partial x_i}(\sigma_{ij}\,\delta u_j) \to \cancel{\frac{\partial \sigma_{ij}}{\partial x_i}\delta u_j} + \sigma_{ij}\frac{\partial \delta u_j}{\partial x_i}$$

$$\rightleftarrows \int_V \sigma_{ij}\frac{\partial \delta u_j}{\partial x_i}\, dV - \int_V \rho k_j\, \delta u_j\, dV - \int_{S_t} t_j\, \delta u_j\, dS = 0$$

$$\frac{1}{2}(\sigma_{ij} + \sigma_{ji})\frac{\partial \delta u_j}{\partial x_i} = \boxed{\frac{1}{2}}\,\sigma_{ij}\,\underbrace{\boxed{\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right)}}_{\delta\varepsilon_{ij}} = \sigma_{ij}\,\delta\varepsilon_{ij}$$

$$\rightleftarrows \boxed{\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\, dV - \int_V \rho k_j\, \delta u_j\, dV - \int_{S_t} t_j\, \delta u_j\, dS = 0}$$

成分として書き出す

$$\int_V (\sigma_{11}\,\delta\varepsilon_{11} + \sigma_{22}\,\delta\varepsilon_{22} + \sigma_{33}\,\delta\varepsilon_{33} + \sigma_{23}\,\delta\gamma_{12} + \sigma_{31}\,\gamma_{31} + \sigma_{12}\,\delta\gamma_{12})\,dV$$

$$- \int_V (\rho k_1\,\delta u_1 + \rho k_2\,\delta u_2 + \rho k_3\,\delta u_3) - \int_{S_t} (\underline{t}_1\,\delta u_1 + \underline{t}_2\,\delta u_2 + \underline{t}_3\,\delta u_3)\,d$$

$$= 0$$

$P \quad \ell$

$t_1 A\,\delta u_1|_{x=\ell}$

（一次元）← 仮想仕事の式

$$\int_V \sigma_{11}\,\delta\varepsilon_{11}\,dV - \int_V \underbrace{\rho k_1\,\delta u_1}_{\text{体積力}}\,dV - \int_{S_t} t_1\,\delta u_1\,ds = 0$$

（棒の問題）　$k_1 = 0$, $u_1 = u$, $\delta u_1 = \delta u$, $x_1 = x$

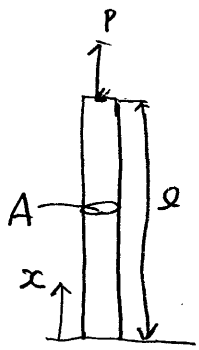
未定定数

$u = C_1 x$

$\delta u = x$

$$A \int_0^{\ell} \underbrace{\left(E\frac{du}{dx}\right)}_{EC_1} \cdot \underbrace{\left(\frac{\partial \delta u}{\partial x}\right)}_{1} dx - \boxed{P\ell} = 0$$

$x=\ell$ を δu_1 に代入



$\rightleftarrows \quad EA\cancel{\ell}C_1 - P\cancel{\ell} = 0$

$\rightleftarrows \quad C_1 = \dfrac{P}{EA} = \varepsilon_{11}$

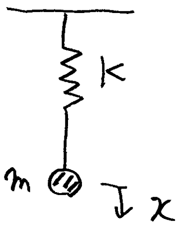
代入

$\rightleftarrows \quad u = \dfrac{Px}{EA}$

$$\sigma_{11} = E\frac{du_1}{dx_1} = EC_1$$

ポテンシャルエネルギ最小の定理

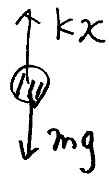
(ばね) $\Pi = \frac{1}{2}kx^2 - mgx$



$$\frac{\partial \Pi}{\partial x} = kx - mg = 0$$

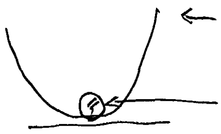
$$\rightarrow x = \frac{mg}{k} \quad (\text{停留条件})$$

自由物体平衡



$\Rightarrow x = \frac{mg}{k}$

$$\frac{\partial^2 \Pi}{\partial x^2} = k > 0 \quad (\text{最小条件})$$



$\frac{\partial^2 \Pi}{\partial x^2} > 0$

(固体)

$$\Pi = \int_V \underbrace{\frac{1}{2}\sigma_{ij}\varepsilon_{ij}}_{\text{ひずみエネルギー}} dV - \int_V \rho k_j u_j dV - \int_{S_t} t_j u_j dS$$

$\sigma_{12}\varepsilon_{12} + \sigma_{21}\varepsilon_{21}$ → γ_{12}

等しい

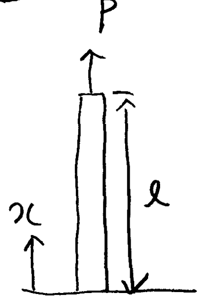
$$\Pi = \int_V \frac{1}{2}(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + \sigma_{12}\gamma_{12} + \sigma_{13}\gamma_{13} + \sigma_{23}\gamma_{23})dV$$

$$- \int_V (\rho k_1 u_1 + \rho k_2 u_2 + \rho k_3 u_3) dV - \int_{S_t} (t_1 u_1 + t_2 u_2 + t_3 u_3) dS$$

(一次元)

$$\Pi = \int_V \frac{1}{2}\sigma_{11}\varepsilon_{11} dV - \int_V \rho k_1 u_1 dV - \int_{S_t} t_1 u_1 dS$$

$\int_{S_t} t_1 u_1 dS \rightarrow Pcl$



$u = cx$

$\varepsilon = c$

$\sigma = Ec$

$$\rightarrow \Pi = A\int_0^l \frac{1}{2}(Ec)\cdot c \, dx - Pcl$$

$$= \frac{1}{2}EAlc^2 - Pcl$$

$$\rightarrow \frac{\partial \Pi}{\partial c} = 0 \Rightarrow EAlc - Pl = 0 \Rightarrow c = \frac{P}{EA}$$

$$\rightarrow \frac{\partial^2 \Pi}{\partial c^2} = \underbrace{EAl}_{} > 0$$

$$\begin{cases} u = \frac{Px}{EA} \\ \sigma = \frac{P}{EA} \times E = \frac{P}{A} \end{cases}$$

行列表示する

$$\Pi = \int_V \frac{1}{2} \underbrace{\begin{pmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \gamma_{23} & \gamma_{31} & \gamma_{12} \end{pmatrix}}_{\boldsymbol{\varepsilon}^T} \underbrace{\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix}}_{\boldsymbol{\sigma}} dV$$

$$- \int_V \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \rho k_1 \\ \rho k_2 \\ \rho k_3 \end{pmatrix} dV - \int_{S_t} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \underline{t}_2 \\ \underline{t}_3 \end{pmatrix} dS$$

$$\boldsymbol{\sigma} = \mathbb{D} \boldsymbol{\varepsilon} \Rightarrow \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix} = \underbrace{[D_{ij}]}_{\text{行列}} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix}$$

$$[D_{ij}] = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \left[\begin{array}{ccc|ccc} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & & & \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & & \mathbb{O} & \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & & & \\ \hline & & & \frac{1-2\nu}{2(1-\nu)} & & \\ & \mathbb{O} & & & \frac{1-2\nu}{2(1-\nu)} & \\ & & & & & \frac{1-2\nu}{2(1-\nu)} \end{array} \right]$$

$$\longrightarrow W = \frac{1}{2} \boldsymbol{\varepsilon}^T \mathbb{D} \boldsymbol{\varepsilon} \geq 0$$

正定値対称行列

$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \underset{\text{重み}}{\alpha} \underbrace{\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}}_{\text{じょう乱}}$$

$$\varepsilon_{ij}^{*} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij}} + \alpha \cdot \underbrace{\frac{1}{2}\left(\frac{\partial \eta_i}{\partial x_j} + \frac{\partial \eta_j}{\partial x_i}\right)}_{\hat{\varepsilon}_{ij}}$$

$$\Pi = \int_V \frac{1}{2} \left(\varepsilon_{11} + \alpha\hat{\varepsilon}_{11} \quad \varepsilon_{22} + \alpha\hat{\varepsilon}_{22} \quad \varepsilon_{33} + \alpha\hat{\varepsilon}_{33} \quad \gamma_{23} + \alpha\hat{\gamma}_{23} \quad \gamma_{31} + \alpha\hat{\gamma}_{31} \quad \gamma_{12} + \alpha\hat{\gamma}_{12} \right)$$

$$[D_{ij}] \begin{pmatrix} \varepsilon_{11} + \alpha\hat{\varepsilon}_{11} \\ \varepsilon_{22} + \alpha\hat{\varepsilon}_{22} \\ \varepsilon_{33} + \alpha\hat{\varepsilon}_{33} \\ \gamma_{23} + \alpha\hat{\gamma}_{23} \\ \gamma_{31} + \alpha\hat{\gamma}_{31} \\ \gamma_{12} + \alpha\hat{\gamma}_{12} \end{pmatrix} dV$$

$$- \int_V \left(u_1 + \alpha\eta_1 \quad u_2 + \alpha\eta_2 \quad u_3 + \alpha\eta_3 \right) \begin{pmatrix} \rho k_1 \\ \rho k_2 \\ \rho k_3 \end{pmatrix} dV$$

$$- \int_{S_t} \left(u_1 + \alpha\eta_1 \quad u_2 + \alpha\eta_2 \quad u_3 + \alpha\eta_3 \right) \begin{pmatrix} \underset{\sim}{t}_1 \\ \underset{\sim}{t}_2 \\ \underset{\sim}{t}_3 \end{pmatrix} dS$$

$$\delta\Pi \equiv \left(\frac{d\Pi(\bar{u}_i)}{d\alpha} \right) \Bigg|_{\alpha=0} \alpha$$

(第一変分)

$$\delta\Pi = \int_V \underbrace{\left(\alpha\hat{\varepsilon}_{11} \quad \alpha\hat{\varepsilon}_{22} \quad \alpha\hat{\varepsilon}_{33} \quad \alpha\hat{\gamma}_{23} \quad \alpha\hat{\gamma}_{31} \quad \alpha\hat{\gamma}_{12} \right)}_{\delta\varepsilon_{ij}} \underbrace{[D_{ij}] \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix}}_{\sigma_{ij}} dV$$

$$-\int_V \underbrace{(\alpha\eta_1\ \ \alpha\eta_2\ \ \alpha\eta_3)}_{\delta u_j} \underbrace{\begin{pmatrix}\rho k_1\\ \rho k_2\\ \rho k_3\end{pmatrix}}_{\rho k_j} dV$$

$$-\int_{S_t} \underbrace{(\alpha\eta_1\ \ \alpha\eta_2\ \ \alpha\eta_3)}_{\delta u_j} \underbrace{\begin{pmatrix}\underline{t}_1\\ \underline{t}_2\\ \underline{t}_3\end{pmatrix}}_{\underline{t}_j} dS$$

$$= \boxed{\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV - \int_V \delta u_j\,\rho k_j\,dV - \int_{S_t} \delta u_j\,\underline{t}_j\,dS = 0}$$

仮想仕事の式

$\delta\Pi = 0$

$$\delta^2\Pi = \frac{1}{2}\left.\frac{d^2\Pi}{d\alpha^2}\right|_{\alpha=0}\alpha^2 \qquad (\text{オ}=\text{変分})$$

$$\delta^2\Pi = \int_V \frac{1}{2}(\alpha\hat{\varepsilon}_{11}\ \ \alpha\hat{\varepsilon}_{22}\ \ \alpha\hat{\varepsilon}_{33}\ \ \alpha\hat{\gamma}_{23}\ \ \alpha\hat{\gamma}_{31}\ \ \alpha\hat{\gamma}_{12})\,[D_{ij}] \begin{pmatrix}\alpha\hat{\varepsilon}_{11}\\ \alpha\hat{\varepsilon}_{22}\\ \alpha\hat{\varepsilon}_{33}\\ \alpha\hat{\gamma}_{23}\\ \alpha\hat{\gamma}_{31}\\ \alpha\hat{\gamma}_{12}\end{pmatrix} dV \geqq 0$$

正定値
対称

安定最小